Phys 410
Fall 2015
Lecture #5 Summary
15 September, 2015

Energy is an abstract theoretical concept that is not associated with any physical entity or mechanism. It is a useful quantity to keep track of, and physicists in some sense are just bookkeepers of energy. Energy comes in many forms. We first encounter kinetic energy $T = \frac{1}{2}mv^2$ for a single particle. The kinetic energy can change when the particle is acted upon by a force that has a component along the direction of displacement of the particle: $dT = \vec{F} \cdot d\vec{r}$. This leads to the Work-Kinetic energy theorem: $T_2 - T_1 = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(r') \cdot d\vec{r}'$, where the value of the line integral (known as 'work') will in general depend on the path (or contour) taken between the points $\vec{r}_1$ and $\vec{r}_2$.

There are two types of forces – conservative and non-conservative. Conservative forces have potential energy functions associated with them. To be conservative, a force must satisfy two requirements:

1) The force depends only on the particle coordinates, and not the velocity, momentum, time, etc.
2) The work done by the force between any two points must be independent of path.

Examples of conservative forces include gravity and the electrostatic force. Non-conservative forces include friction and the drag forces that we considered earlier.

The potential energy is defined as follows. Choose an arbitrary position $\vec{r}_0$ where the potential energy will be assigned a value of 0. The potential energy is defined in terms of the work done on the particle to take it from $\vec{r}_0$ to any point $\vec{r}$: $U(\vec{r}) \equiv -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(r') \cdot d\vec{r}'$, where there is no need to specify the contour for the line integral. Note the minus sign. With this definition, one can show that if only conservative forces act, the total mechanical energy $E = T + U$ of the system is conserved, i.e. $\Delta E = 0$. This conservation law is very useful for solving problems. If non-conservative forces do act, along with conservative forces, then the mechanical energy of the system changes by an amount equal to the work done by the non-conservative forces: $\Delta E = W_{nc}$. W_{nc} is typically negative because non-conservative forces usually 'steal' mechanical energy and convert it to heat (thermal energy).

We considered the process of deducing a vector force from a given scalar potential-energy function. This is done through the gradient differential operator $\vec{F} = -\vec{\nabla}U$. Note that this is actually three equations in one. We did the example of the electrostatic potential $U = kq_1q_2/r$, and showed that the associated force is the Coulomb electrostatic force $\vec{F} =$

$kq_1q_1\hat{r}/r^2$. Since conservative forces are derived from a potential energy function, we can find a simple necessary (but not sufficient) test to see if the force really is conservative. Taking the curl of a conservative force yields $\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times \vec{\nabla} U = 0$, where the last term is a vector identity good for all (well-behaved) scalar functions $U(\vec{r})$. Hence all conservative forces must be curl-free vector fields. An additional requirement is that the force depends only on the particle coordinates.

We considered energy for motion in one-dimensional systems. This is not as artificial as it first appears – later we will be able to break certain 3D problems to simpler 2D and 1D problems, and the methods that follow will be very useful. Consider a particle confined to move only on the x-axis. It has a kinetic energy $T = \frac{1}{2} m\dot{x}^2$. The kinetic energy can be altered by applying a force and doing work on the particle. The tangential component of the force does work as $W(x_0 \to x_1) = \int_{x_0}^{x_1} F_{\tan}(x')\, dx'$. If this force is conservative, one can define an associated potential energy (PE) as $U(x) = -W(x_0 \to x) = -\int_{x_0}^{x} F_{\tan}(x')\, dx'$, where it is assumed that $U(x_0) = 0$. We also expect that the total mechanical energy will be conserved: $E = T + U(x)$, and $\Delta E = 0$. This conservation law allows elegant solution of 1D problems involving conservative forces.

We did an example of a Hooke's law restoring force in 1D: $\vec{F} = -k\, x\, \hat{x}$, with an equilibrium point at $x = 0$. The corresponding potential is $U(x) = \frac{1}{2} kx^2$, with $U(0) = 0$. The mechanical energy is conserved: $E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$. As the particle moves it exchanges energy back and forth between kinetic and potential energies.

Consider a particle moving under just the Hooke's law restoring force in one dimension. We call this a harmonic oscillator. If the particle has a total mechanical energy E, it will move back and forth between two values of x, call them $\pm x_{max}$. As the particle approaches the extreme value of x it will be at a point where $U = E$, hence it must be the case that $T = 0$, and the particle comes to rest. This is called the classical turning point. Classically the particle reverses direction and does not venture beyond the classical turning point because it would require a negative value for T, which is not possible classically. However in quantum mechanics the kinetic energy operator in one-dimensional wave mechanics is given by $T = \frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2}$, which is basically minus the curvature of the wavefunction. In this case both signs of the kinetic energy are possible. For locations inside the classical turning points, the wavefunction is concave towards the axis, meaning negative curvature and a positive kinetic energy. For locations "outside the well," meaning beyond the classical turning points, the wavefunction is concave away from the axis, meaning positive curvature and negative kinetic energy. Either way, mechanical energy $E = T + U(x)$ is conserved in the motion.